Friday, May 20, 2016 11:07 AM

1) The paraboloid $z = 4-x^2-y^2$ intersects the xy-plane when $0 = 4-x^2-y^2 \Rightarrow x^2+y^2=4$
$\Rightarrow r=2$.

Then in cylindrical coordinate

$$E = \{(r,\theta,z) \mid 0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le z \le 4-r^2\}$$

Therefore, $\iiint_E x+y+z\, dV = \int_0^{\pi/2}\int_0^{2}\int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\, r\, dr\, d\theta\, dz$

$$= \int_0^{\pi/2}\int_0^{2} r^2(\cos\theta+\sin\theta)z + \frac{1}{2} r z^2 \Big]_{z=0}^{4-r^2} dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{2} \left[(4r^2-r^4)(\cos\theta+\sin\theta) + \frac{1}{2} r(4-r^2)^2\right] dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\left(\frac{4}{3}r^3 - \frac{1}{5}r^4\right)(\cos\theta+\sin\theta) - \frac{1}{4}(4-r)^3\right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{\pi/2} \left(\frac{64}{15}(\cos\theta+\sin\theta) + \frac{16}{3}\right) d\theta = \left[\frac{64}{15}(\sin\theta - \cos\theta) + \frac{16}{3}\theta\right]_0^{\pi/2}$$

$$= \frac{64}{15}(1-0) + \frac{16}{3}\cdot\frac{\pi}{2} - \frac{64}{15}(0-1) - 0 = \frac{128}{15}\pi + \frac{8}{3}\pi.$$

2) The paraboloid $z = 4x^2+4y^2$ intersects the plane $z=a$ at $4x^2+4y^2=a$
$\Rightarrow x^2+y^2 = \left(\frac{\sqrt{a}}{2}\right)^2$

Therefore in cylindrical coordinates, E is given by

$$E = \{(r,\theta,z) \mid 0 \le r \le \frac{\sqrt{a}}{2},\ 0 \le \theta \le \pi,\ 4r^2 \le z \le a\}$$

Then, $m = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} K r\, dz\, dr\, d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2} rz\Big]_{4r^2}^{a} dr\, d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2} (ar - 4r^3)\, dr\, d\theta$

Monday, May 23, 2016 8:28 AM

$$= K\int_0^{2\pi}\left[\frac{1}{2}ar^2 - r^4\right]_0^{\sqrt{a/2}} d\theta = K\int_0^{2\pi}\frac{1}{2}\frac{a^2}{4} - \frac{a^2}{16}\,d\theta = K\int_0^{2\pi}\frac{1}{16}a^2\,d\theta = \frac{1}{8}K\pi a^2$$

- The region in question is symmetric about the yz-plane and xz-plane and so is the density function (as it is constant). Therefore,

$M_{yz} = M_{xz} = 0$

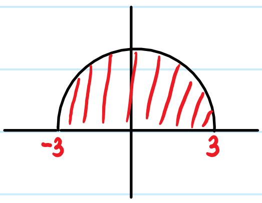
$$M_{xy} = \int_0^{2\pi}\int_0^{\sqrt{a/2}}\int_{4r^2}^{a} Krz\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a/2}}\left[r\frac{z^2}{2}\right]_{4r^2}^{a} dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a/2}}\frac{1}{2}a^2r - 8r^5\,dr\,d\theta$$

$$= K\int_0^{2\pi}\left[\frac{1}{4}a^2r^2 - \frac{4}{3}r^6\right]_0^{\sqrt{a/2}} d\theta = K\int_0^{2\pi}\left[\frac{1}{16}a^3 - \frac{1}{48}a^3\right] d\theta = K\int_0^{2\pi}\frac{1}{24}a^3\,d\theta = \frac{1}{12}a^3\pi K$$

Then, $(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(0, 0, \frac{2}{3}a\right)$.

3) The region of integration in question can be described as follows:

- It is above the plane $z = 0$ and below the paraboloid $z = 9 - x^2 - y^2$.
- Also, $-3 \le x \le 3$ w/ $0 \le y \le \sqrt{9-x^2}$ means the region



Thus region of integration in cylindrical coordinates given by

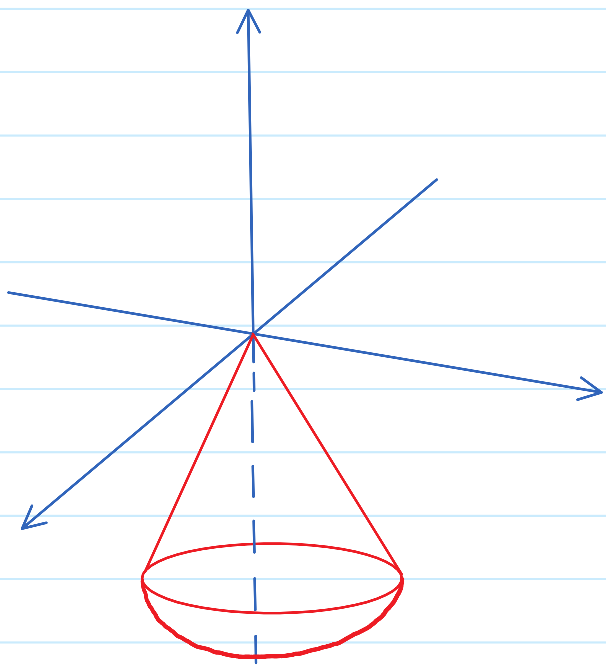
$$E = \{(r,\theta,z) \mid 0 \le \theta \le \pi,\ 0 \le r \le 3,\ 0 \le z \le 9 - r^2\}$$

So, $$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\,dy\,dz = \int_0^{\pi}\int_0^{3}\int_0^{9-r^2}\sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{3}\int_0^{9-r^2} r^2\,dz\,dr\,d\theta$$

Monday, May 23, 2016 8:47 AM

$$= \int_0^{\pi}\int_0^3 r^2(9-r^2)\,dr\,d\theta = \int_0^{\pi} d\theta \int_0^3 9r^2 - r^4\,dr = \pi\left[3r^3 - \frac{r^5}{5}\right]_0^3 = \pi\left(81 - \frac{241}{5}\right) = \frac{162}{5}\pi .$$

4) $\rho \le 1$ represents the solid sphere of radius 1 centered at the origin.
$\frac{3\pi}{4} \le \phi \le \pi$ restricts to the portion on & below the cone given by $\phi = \frac{3\pi}{4}$.



5) $E = \{(r,\theta,z) \mid 0 \le \theta \le \pi/2,\ 0 \le r \le 3,\ 0 \le z \le 2\}$.

Then, $\iiint_E f(x,y,z)\,dV = \int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$

6) $E = \{(\rho,\theta,\phi) \mid 2 \le \rho \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$

- $x^2+y^2 = r^2 = \rho^2\sin^2\phi$

Then, $\iiint_E x^2+y^2\,dV = \int_0^{\pi}\int_0^{2\pi}\int_2^3 \rho^2\sin^2\phi\ \rho\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^{\pi}\sin^3\phi\,d\phi \int_0^{2\pi} d\theta \int_2^3 \rho^4\,d\rho$

Monday, May 23, 2016 9:02 AM

$$= \int_0^{\pi} \sin\phi\,(1-\cos^2\phi)\,d\phi \cdot \left[2\pi\right] \cdot \left[\frac{\rho^5}{5}\right]_2^3$$

$$= \left[\int_0^{\pi} \sin\phi\,d\phi - \int_0^{\pi} \sin\phi\cos^2\phi\,d\pi\right] \cdot 2\pi \cdot \frac{211}{5}$$

$u = \cos\phi$

$$= \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi} \cdot \frac{422\pi}{5} = \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) \cdot \frac{422\pi}{5} = \frac{1688\pi}{15}.$$

7) The cone $z = \sqrt{x^2+y^2}$ is represented by $\phi = \pi/4$, and below the cone and above the xy-plane corresponds to $\pi/4 \le \phi \le \pi/2$.

Thus the solid in question E is given by

$$E = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ \pi/4 \le \phi \le \pi/2\}$$

Then,

$$\text{vol}(E) = \iiint_E 1\,dV = \int_{\frac{\pi}{4}}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/4}^{\pi/2} \sin\phi\,d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2\,d\rho$$

$$= \left[-\cos\phi\right]_{\pi/4}^{\pi/2} \cdot 2\pi \cdot \left[\frac{\rho^3}{3}\right]_0^2 = \frac{\sqrt{2}}{2} \cdot 2\pi \cdot \frac{8}{3} = \frac{8\sqrt{2}\,\pi}{3}$$

8) The transformation T is given by

$x(r,\theta,z) = r\cos\theta$, $z(r,\theta,z) = z$.

$y(r,\theta,z) = r\sin\theta$

The Jacobian is given by

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Then, $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = |r| = r$ (since $r > 0$).

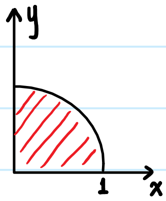
Then,

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

9) The region of integration E is described as follows:

$\sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}$ means the region is above the cone $z = \sqrt{x^2+y^2}$ $[\phi = \pi/4]$ and below the surface $z = \sqrt{2-x^2-y^2}$ which is $x^2+y^2+z^2 = 2$ $[\rho = \sqrt{2}]$.

Then, the region $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$ is given by



Thus, $E = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 2,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le \pi/4\}$

Therefore,

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}} (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

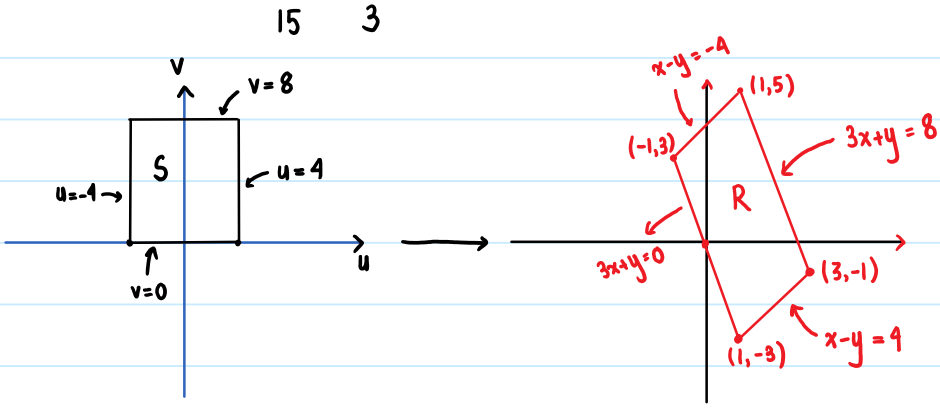
$$= \int_0^{\pi/4} \sin^3\phi\,d\phi \int_0^{\pi/2} \sin\theta\cos\theta \int_0^{\sqrt{2}} \rho^4\,d\rho$$

$$= \left(\int_0^{\pi/4} (1-\cos^2\phi)\sin\phi\right) \cdot \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} \cdot \left[\frac{1}{5}\rho^5\right]_0^{\sqrt{2}}$$

$$= \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi/4} \cdot \frac{1}{2} \cdot \frac{4\sqrt{2}}{5} = \left[\frac{1}{3}\cdot\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2} - \frac{1}{3} + 1\right] \cdot \frac{2\sqrt{2}}{5}$$

$$= \frac{4\sqrt{2}}{15} - \frac{1}{3}$$

10)



6

Monday, May 23, 2016 10:04 AM

R is the parallelogram bounded by lines

$x-y=-4$; $3x+y=0$

$x-y=4$; $3x+y=8$

• $x-y = \frac{1}{4}u + \frac{1}{4}v - \frac{1}{4}v + \frac{3}{4}u = u$ & $3x+y = \frac{3}{4}u + \frac{3}{4}v + \frac{1}{4}v - \frac{3}{4}u = v$

So the parallelogram R is the image of the rectangle S enclosed by the lines $u=-4$, $u=4$, $v=0$ and $v=8$.

• $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = \frac{1}{16} - \left(\frac{-3}{16}\right) = \frac{4}{16} = \frac{1}{4}$$

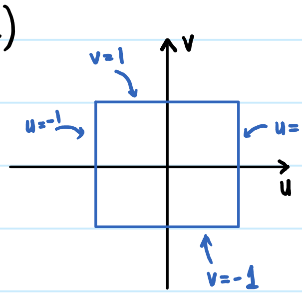
$$4x+8y = 4.\frac{1}{4}(u+v) + 8.\frac{1}{4}(v-3u) = u+v+2u-6v = 3u-5v$$

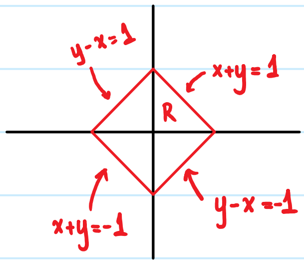
Then,

$$\iint_R (4x+8y)\,dA = \int_{-4}^{4}\int_{0}^{8} (3v-5u)\left|\frac{1}{4}\right| dv\,du = \frac{1}{4}\int_{-4}^{4}\left[3v^2-5uv\right]_{v=0}^{8} = \frac{1}{4}\int_{-4}^{4}(96-40u)\,du$$

$$= \frac{1}{4}\left[96u-20u^2\right]_{-4}^{4} = \frac{1}{4}\left[96.4-20(4)^2-96.(-4)+20.(-4)^2\right] = \frac{1}{4}(96.8) = 192.$$

12)





Once you draw the region R, you see that it is bounded by the lines $x+y=\pm 1$ and $y-x=\pm 1$.

Set,

$u = x+y$, $v = y-x$ $\Rightarrow y = \frac{1}{2}(u+v)$ & $x = \frac{1}{2}(u-v)$.

Then $x+y=\pm 1$ are the image of the lines $u=\pm 1$ and $y-x=\pm 1$ of $v=\pm 1$

Then, $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$

So,

$$\iint_R e^{x+y}\, dA = \int_{-1}^{1}\int_{-1}^{1} e^{u} \left|\frac{1}{2}\right| du\, dv = \frac{1}{2}\int_{-1}^{1} e^{u} du \int_{-1}^{1} 1\, dv = \frac{1}{2}\left[e^{u}\right]_{-1}^{1} . 2 = e - e^{-1}$$

13) $f(x,y) = \sin^2(mx+ny) = [\sin(mx+ny)]^2$

$f_x = 2\sin(mx+ny).\cos(mx+ny).m$; $f_y = 2\sin(mx+ny).\cos(mx+ny).n$

$= 2m\sin(mx+ny).\cos(mx+ny)$ $\quad = 2n\sin(mx+ny).\cos(mx+ny)$

$$f_{yy} = 2n\left[\cos(mx+ny).n.\cos(mx+ny) + \sin(mx+ny).-\sin(mx+ny).n\right]$$

$$= 2n^2\left[\cos^2(mx+ny) - \sin^2(mx+ny)\right]$$

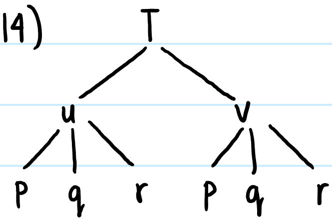
$$f_{xx} = 2m\left[\cos(mx+ny).m.\cos(mx+ny) + \sin(mx+ny).-\sin(mx+ny).m\right]$$

$$= 2m^2\left[\cos^2(mx+ny) - \sin^2(mx+ny)\right]$$

$$f_{xy} = 2m\left[\cos(mx+ny).n.\cos(mx+ny) + \sin(mx+ny).-\sin(mx+ny).n\right]$$

$$= 2mn\left[\cos^2(mx+ny) - \sin^2(mx+ny)\right] = f_{yx}$$

14)



$$\frac{\partial T}{\partial u} = \frac{(2u+v).0 - v(2)}{(2u+v)^2} = \frac{-2v}{(2u+v)^2}$$

$$\frac{\partial T}{\partial v} = \frac{(2u+v).1 - v(1)}{(2u+v)^2} = \frac{2u}{(2u+v)^2}$$

$\dfrac{\partial u}{\partial p} = q\sqrt{r}$; $\dfrac{\partial u}{\partial q} = p\sqrt{r}$; $\dfrac{\partial u}{\partial r} = \dfrac{pq}{2\sqrt{r}}$

$\dfrac{\partial v}{\partial p} = \sqrt{q}\,r$; $\dfrac{\partial v}{\partial q} = \dfrac{pr}{2\sqrt{q}}$; $\dfrac{\partial v}{\partial r} = p\sqrt{q}$

Monday, May 23, 2016 10:44 AM

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial u} \cdot \frac{\partial u}{\partial p} + \frac{\partial T}{\partial v} \cdot \frac{\partial v}{\partial p}$$

$$= \frac{-2v}{(2u+v)^2} \cdot q\sqrt{r} + \frac{2u}{(2u+v)^2} \cdot p\sqrt{r}$$

$$= \frac{-2p\sqrt{q}r}{(2pq\sqrt{r} + p\sqrt{q}r)^2} \cdot q\sqrt{r} + \frac{2pq\sqrt{r}}{(2pq\sqrt{r} + p\sqrt{q}r)^2} \cdot p\sqrt{r}$$

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial u} \cdot \frac{\partial u}{\partial q} + \frac{\partial T}{\partial v} \cdot \frac{\partial v}{\partial q}$$

$$= \frac{-2p\sqrt{q}r}{(2pq\sqrt{r} + p\sqrt{q}r)^2} \cdot p\sqrt{r} + \frac{2pq\sqrt{r}}{(2pq\sqrt{r} + p\sqrt{q}r)^2} \cdot \frac{pr}{2\sqrt{q}}$$

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial u} \cdot \frac{\partial u}{\partial r} + \frac{\partial T}{\partial v} \cdot \frac{\partial v}{\partial r}$$

$$= \frac{-2p\sqrt{q}r}{(2pq\sqrt{r} + p\sqrt{q}r)^2} \cdot \frac{pq}{2\sqrt{r}} + \frac{2pq\sqrt{r}}{(2pq\sqrt{r} + p\sqrt{q}r)^2} \cdot p\sqrt{q}$$

15) $g(r,s) = \arctan(rs)$; $(1,2)$, $\vec{v} = 5\hat{i} + 10\hat{j}$

$$\nabla g(r,s) = \frac{\partial g}{\partial r}\hat{i} + \frac{\partial g}{\partial s}\hat{j}$$

$$= \left\langle \frac{1}{1+(rs)^2} \cdot s, \frac{1}{1+(rs)^2} \cdot r \right\rangle = \left\langle \frac{s}{1+(rs)^2}, \frac{r}{1+(rs)^2} \right\rangle$$

$$\nabla g(1,2) = \left\langle \frac{2}{5}, \frac{1}{5} \right\rangle$$

Note, $\vec{v} = 5\hat{i} + 10\hat{j}$ is not a unit vector, but $|\vec{v}| = \sqrt{5^2+10^2} = \sqrt{150} = 5\sqrt{6}$ and the unit vector in the direction of $\vec{v}$ is given by :

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{5\hat{i} + 10\hat{j}}{5\sqrt{6}} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle$$

Then,

$$D_{\vec{v}}\, g(1,2) = \nabla g(1,2) \cdot \vec{u} = \left\langle \frac{2}{5}, \frac{1}{5} \right\rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle$$

$$= \frac{2}{5\sqrt{6}} + \frac{2}{5\sqrt{6}} = \frac{4}{5\sqrt{6}} = \frac{2\sqrt{6}}{15}.$$

16) $f(x,y,z) = x \ln(y-2z)$

$$\frac{\partial f}{\partial x} = \ln(y-2z), \quad \frac{\partial f}{\partial y} = \frac{x}{y-2z}, \quad \frac{\partial f}{\partial z} = \frac{x}{y-2z} \cdot -2$$

Then,

$$\nabla f(x,y,z) = \ln(y-2z)\,\hat{i} + \frac{x}{y-2z}\,\hat{j} + \frac{-2x}{y-2z}\,\hat{k}.$$

17) $f(x,y) = x^2 - y$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -1, \quad \nabla f(x,y) = 2x\,\hat{i} - \hat{j}$$

The length of $\nabla f(x,y) = \sqrt{(2x)^2 + (-1)^2} = \sqrt{4x^2 + 1}$.

So at each point when $x = 0$, $\nabla f(0,y) = \langle 0,-1 \rangle$ meaning we get a unit vector $\langle 0,-1 \rangle$
When $x \neq 0$, the vectors point away from the y-axis in a slightly downward direction with the length increasing as $|x|$ increases.

Tuesday, May 24, 2016 9:51 AM